MULTIMEDIA UNIVERSITY

STUDENT ID NO

MULTIMEDIA UNIVERSITY

FINAL EXAMINATION

TRIMESTER 2, 2016/2017

EME3156-COMPUTATIONAL METHODS FOR MECHANICAL ENGINEERING

(ME)

9 MARCH 2017
2.30 p.m. – 4.30 p.m.
(2 Hours)

INSTRUCTION TO STUDENT

1. This question paper consists of 9 pages excluding the cover page.
2. This paper is divided into **Sections A and B**. Answer all questions in Section A. Choose two questions to answer in Section B. The distribution of the marks for each question is given.
3. Please write all your answers in the answer booklet provided.
4. All necessary workings **MUST** be shown.

Section A

Question 1

Equation (1.1) shows a first order ODE and the initial condition.

$$\frac{dy}{dx} = 2x^2, \; y(0) = 3 \tag{1.1}$$

a) Show that the analytical solution of the above equation is: $y = \frac{2}{3}x^3 + 3$.

[5 marks]

b) Using the explicit Euler method with step size $h = 0.25$, find $y(1)$.

[10 marks]

c) Make a sketch to show the concept of the midpoint method. The initial condition and approximated values must be labelled and shown clearly.

[4 marks]

d) Using the midpoint method with step size $h = 1$, find $y(1)$.

[4 marks]

e) Find the exact value of $y(1)$. Then, based on the answer in (b) and (d), state which method gives the closer answer to the exact value.

[2 marks]

Continued...

Question 2

Equation (2.1) shows a non-linear second order ODE and the boundary conditions. The domain is discretised into six nodes equally distanced and then solved by utilizing finite difference method.

$$\frac{d^2y}{dx^2} - 2y^2 = 7x,\ y(0) = 2,\ y(1) = 2 \qquad (2.1)$$

a) Sketch the discretisation with the node numbering in sequence starting from "0" at the left boundary.

[3 marks]

b) Write down the finite difference form of Equation (2.1) in numerical values as much as possible by using the central difference for all terms, and then apply this equation to the discretised domain. The boundary conditions must be incorporated into these equations.

[12 marks]

c) The Newton method for non-linear system of equations is used to solve the equations formed in (b), where these equations are assembled to become a matrix equation $[J]\{\Delta x\} = \{F\}$. Expand this matrix equation to show all elements in terms of the nodes in the discretisation. Appropriate subscripts that are consistent with the node numbering must be used.

[10 marks]

Continued...

Section B

Question 3

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad , 0 \le x \le 1.0 \tag{3.1}$$

Equation (3.1) is a one-dimensional heat equation and it can be used to model temperature distribution measured along a homogeneous rod. The constant α is the ratio of heat conduction to the heat storage. The independent variables are x and t.

A problem is to be modeled using heat equation above. α is taken as 1 while the spatial domain is to be discretized into six equally spaced nodes. Use time interval of $\Delta t = 0.02$ for calculation. The initial and boundary conditions are given by

$$u(x,0) = -\frac{1}{2}\sin 3\pi x + \frac{3}{2}\sin \pi x, \quad 0 \le x \le 1.0$$
$$u(0,t) = u(1.0,t) = 0$$

a) Equation (3.1) is to be solved using explicit finite difference method. Convert equation (3.1) into finite difference form using central difference scheme.

[3 marks]

b) Present the finite difference form (from part (a)) in matrix form, as given by the equation (3.2) below..

$$\{u\}^{t+1} = [A]\{u\}^t + \{\text{Boundary Conditions}\} \tag{3.2}$$

$\{u\}^{t+1}$ and $\{u\}^t$ are vectors for nodal variables at time t+1 and t respectively. $[A]$ is a square matrix.

[4 marks]

c) Solve for vector for nodal variables $\{u\}^1$ and $\{u\}^2$ at time t+1 and t +2 .

[8 marks]

d) State the stability criteria for **implicit** finite difference method. **[2 marks]**

e) The analytical solution is given by $u(x,t) = \frac{3}{2}\sin \pi x e^{-\pi^2 t} - \frac{1}{2}\sin 3\pi x e^{-9\pi^2 t}$

Compare the analytical solution with that from part (c) at t = 0.04. Comment on the accuracy of the numerical calculation. How would you increase the accuracy of your numerical calculation ?

[8 marks]

Continued...

Question 4

Temperature distribution of a two-dimensional irregular metal can be expressed in terms of a steady state diffusion equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{f(x,y)}{a}=0 \qquad (4.1)$$

where $f(x,y)$ is the power density at (x,y) and a is the thermal conductivity. For this problem, $\frac{f(x,y)}{a}=0$

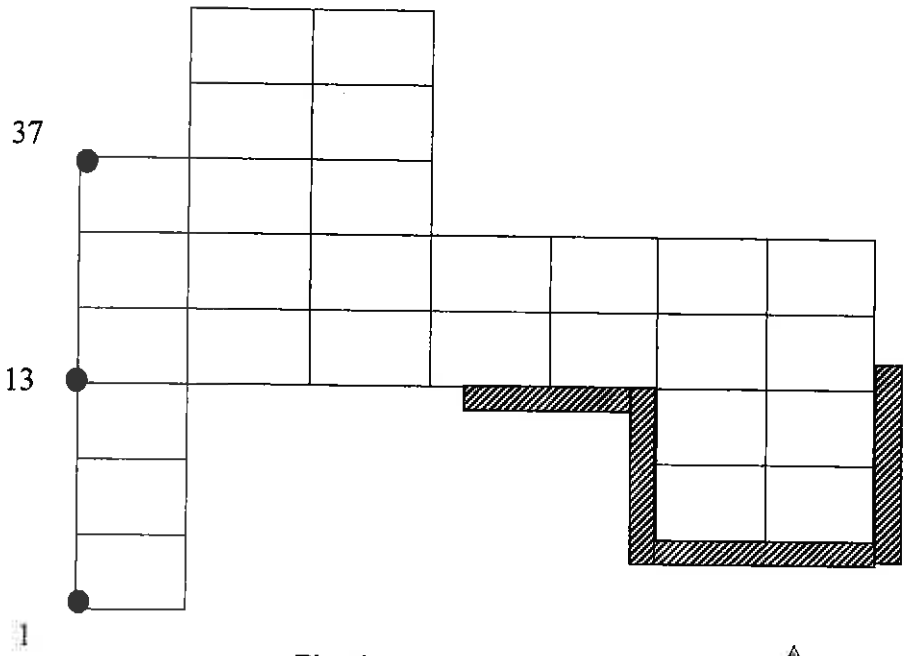

Fig. 4.1

For the highlighted area,

insulated

Node 1, 3, 8, 13, 21, 29, 37, 38, 41 and 44 are at temperature 60°C
Node 14, 15, 16, 32, 33, 34, 35, 40, 43, 45 and 46 are at temperature 57°C
Node 28 and 36 are at temperature 55°C
Node 2, 4, and 9 are at temperature 58°C
Node 5, 6, 7, 10, 12, 17, 18 and 20 are insulated

Continued...

a) Fig. 4.1 above shows the temperature distribution at the boundaries of a metal plate. The metal plate domain consists of 28 rectangles with each of them of the size 2cm x 2cm. The domain above is discretized into equally distanced nodes on each edge on the x-y plane. Node 1 is considered as the origin of the x-y coordinate system and it is labeled in Fig 4.1. Nodes 13 and 37 are also labelled for you. Copy the figure into your answer sheet, and label all the missing nodes. The labeling of the nodes start from left to right and then goes from bottom to top.

[3 marks]

b) By using central difference of the second derivative, find the finite difference form of the equation (4.1).

[9 marks]

c) Rewrite equation (4.1) in matrix form given by:

$$[A]\{u\} = \{c\} \qquad (2.2)$$

where $\{u\}$ is a vector for nodal variable at a two-dimensional location and $\{c\}$ is the vector after incorporating the temperature at the boundaries. $[A]$ is a square matrix. DO NOT ATTEMPT TO SOLVE IT but only reproduce the Laplace equation (4.1) in matrix form.

[13 marks]

Continued...

Question 5:

In a big rectangular tank, a wave is initiated and it progresses without any blockade in forward direction. The wave equation is given by

$$6\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad t \ge 0 \tag{5.1}$$

The velocity, boundary and initial conditions are

$$u_x(0,t) = \frac{\pi}{8}, \; u(1,t) = -0.55, \; u(x,0) = \sin(\pi x)\cos(\pi x), \; \frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{\pi}{4}\right)$$

The partial differential equation above is to be solved using finite difference method. The spatial domain is to be discretized into 6 nodes equally spaced and to calculate the wave progression in time, use $\Delta t = 0.05$.

a) By using central difference of the second derivative, approximate the derivative $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$ in their respective variables. The subscripts should represent spatial location (x) while the superscripts should represent time steps (t).

[4 marks]

b) One or more of the initial conditions given is/are the Neumann Initial Condition/s. Write down the initial condition/s, and the corresponding numerical value/s after the equation/s is/are converted into finite difference form.

[4 marks]

c) Sketch the initial profile, i.e. $u(x,0)$, for $0 \le x \le 1$. Remember the spatial domain is discretized into 6 equally spaced nodes. **[4 marks]**

d) Incorporating initial and boundary conditions into the problem, rewrite equation (5.1) in matrix form,

$$\{u\}^{t+1} = [A]\{u\}^t + \{u\}^{t-1} + \{\text{boundary conditions}\}$$

where $\{u\}^{t+1}$, $\{u\}^t$ and $\{u\}^{t-1}$ are vectors for nodal variables at time t+1, t and t-1 respectively. $[A]$ is a square matrix. **[5 marks]**

e) Evaluate $\{u\}^1$,vector for nodal variables at time t +1.

[8 marks]

Continued...

Appendix

Explicit Euler method

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

Implicit Euler method

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$

Modified Euler method (predictor-corrector)

Step 1: $K_1 = f(x_i, y_i),\ y_{i+1}^P = y_i + hK_1$

Step 2: $K_2 = f(x_{i+1}, y_{i+1}^P),\ y_{i+1}^C = y_i + h\frac{1}{2}(K_1 + K_2)$

Midpoint method

Step 1: $K_1 = f(x_i, y_i),\ y_m = y_i + \frac{h}{2}K_1$

Step 2: $K_2 = f(x_m, y_m),\ y_{i+1} = y_i + hK_2$

Fourth-order Runge-Kutta

Step 1: $K_1 = f(x_i, y_i),\ y_{m1} = y_i + \frac{h}{2}K_1$

Step 2: $K_2 = f(x_m, y_{m1}),\ y_{m2} = y_i + \frac{h}{2}K_2$

Step 3: $K_3 = f(x_m, y_{m2}),\ y_{m3} = y_i + hK_3$

Step 4: $K_4 = f(x_{i+1}, y_{m3}),\ y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Second-order Adams-Bashforth

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

Third-order Adams-Bashforth

$$y_{i+1} = y_i + \frac{h}{12}(23f_i - 16f_{i-1} + 5f_{i-2})$$

Fourth-order Adams-Bashforth

$$y_{i+1} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

Second-order Adams-Moulton

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1})$$

Third-order Adams-Moulton

$$y_{i+1} = y_i + \frac{h}{12}(5f_{i+1} + 8f_i - f_{i-1})$$

Fourth-order Adams-Moulton

$$y_{i+1} = y_i + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2})$$

First derivative, forward difference

Continued...

$$y' = \frac{1}{h}\left(y_{i+1} - y_i\right)$$

First derivative, backward difference

$$y' = \frac{1}{h}\left(y_i - y_{i-1}\right)$$

First derivative, central difference

$$y' = \frac{1}{2h}\left(y_{i+1} - y_{i-1}\right)$$

Second derivative, forward difference

$$y'' = \frac{1}{h^2}\left(y_i - 2y_{i+1} + y_{i+2}\right)$$

Second derivative, backward difference

$$y'' = \frac{1}{h^2}\left(y_{i-2} - 2y_{i-1} + y_i\right)$$

Second derivative, central difference

$$y'' = \frac{1}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right)$$

Jacobi method

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^k\right)$$

Gauss-Seidel method

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k\right)$$

Successive over-relaxation method

$$x_i^{k+1} = (1-\omega)x_i^k + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k\right)$$

Jacobian matrix

$$[J] = \begin{bmatrix} \frac{\partial f_i}{\partial x_1} & \frac{\partial f_i}{\partial x_2} & \cdots & \frac{\partial f_i}{\partial x_n} \\ \frac{\partial g_i}{\partial x_1} & \frac{\partial g_i}{\partial x_2} & \cdots & \frac{\partial g_i}{\partial x_n} \\ \frac{\partial h_i}{\partial x_1} & \frac{\partial h_i}{\partial x_2} & \cdots & \frac{\partial h_i}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

Sherman-Morrison formula

$$[J]_{\text{new}}^{-1} = [J]_{\text{old}}^{-1} + \frac{\left\{\{\Delta x\} - [J]_{\text{old}}^{-1}\{\Delta F\}\right\}\{\Delta x\}^{\text{T}}[J]_{\text{old}}^{-1}}{\{\Delta x\}^{\text{T}}[J]_{\text{old}}^{-1}\{\Delta F\}}$$

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